Section 12.1: Three-Dimensional Coordinate Systems

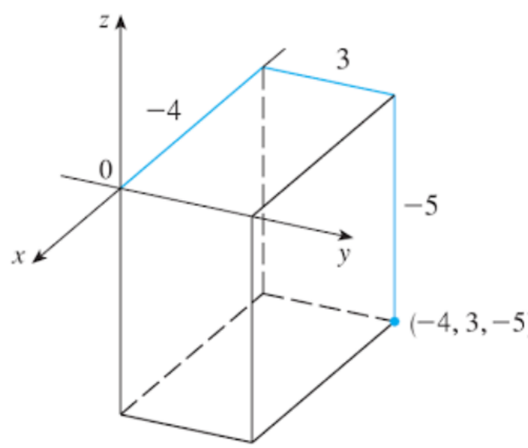
DEF: Points in two-dimensions belong to the set

$$\mathbb{R}^2 = \{(x, y)|\ x, y \in \mathbb{R}\}$$

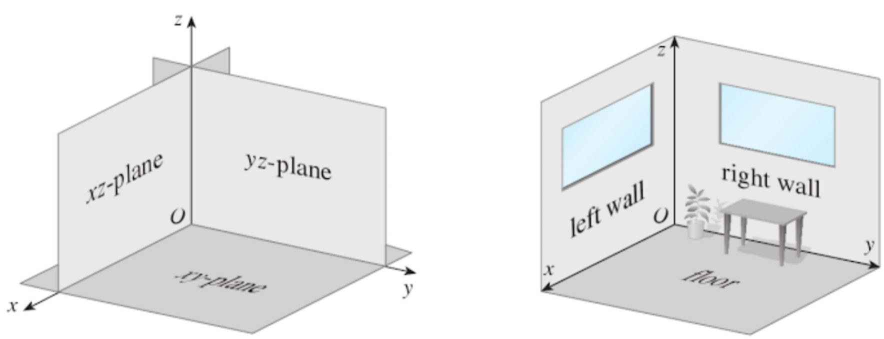
Ex1. Sketch the point $(1, 2)$ in $\mathbb{R}^2$.

DEF: Points in three-dimensions belong to the set $\mathbb{R}^3 = \{(x, y, z)|\ x, y, z \in \mathbb{R}\}$.

Ex2. Sketch the point $(-4, 3, -5)$ in $\mathbb{R}^3$.



DEF: The set of points in $\mathbb{R}^3$ such that $z = 0$ is called the xy-plane.

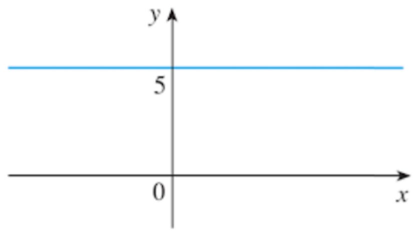


Ex3. What is the (shortest) distance from the point $(-4, 3, -5)$ to the xy-plane.

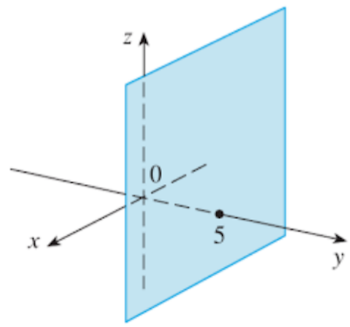
Surfaces
In two dimensional analytic geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional analytic geometry, an equation in x, y, and z represents a surface in $\mathbb{R}^3$.

Ex4. Sketch the set of points contained in:

(a) $\mathbb{R}^2$, satisfying $y = 5$.



(b) $\mathbb{R}^3$, satisfying $y = 5$.



(c) $\mathbb{R}^3$, satisfying $x^2 + y^2 = 4$.

(d) $\mathbb{R}^3$, satisfying $y = x^2 + 1$ and $0 \leq z \leq 4$.

Distance Formula in $\mathbb{R}^3$.
We will denote and define the distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ by

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Ex5. Find the distance $|PQ|$ between the two points $P(2, -1, 7)$ and $Q(1, -3, 5)$.

Using the distance formula we derive an equation for the sphere in $\mathbb{R}^3$ with center $P_0(a, b, c)$ and radius r.

Ex6. Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 5 = 0$ is the equation of a sphere. Find the center and the radius.

Ex7. Sketch and describe the set of points in $\mathbb{R}^3$ satisfying the inequality

$$x^2 + y^2 + z^2 \leq 4y.$$

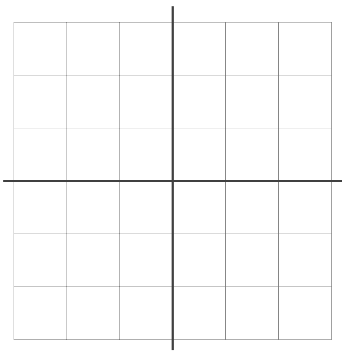
TO-DO: Repeat Ex7 with $>$.

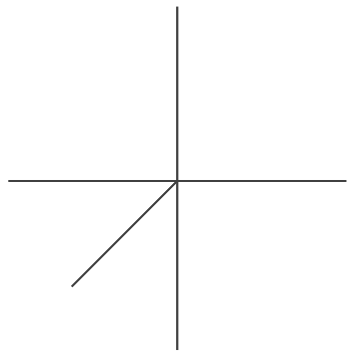
Section 12.2: Vectors.

The term **vector** is used to indicate a quantity that has both magnitude and direction (such as displacement, velocity, force, etc). A vector is often represented by an arrow. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector.

A vector in two dimensions is denoted by $\vec{a} = \langle a_1, a_2 \rangle$. The numbers a_1 and a_2 are called the components of the vector. Similarly, the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ in three dimensions has components a_1, a_2 and a_3.

Ex1. Sketch the vectors $\vec{a} = \langle 2, 1 \rangle$, $\vec{b} = \langle 1, -1 \rangle$, and $\vec{c} = \langle 1, 2, 3 \rangle$.





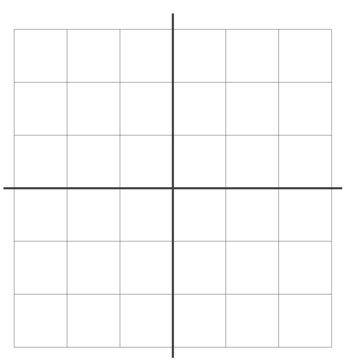
Given points $A(x_1, y_1)$ and $B(x_2, y_2)$ the displacement vector $\overrightarrow{AB} = \langle a_1, a_2 \rangle$ is defined so that $x_1 + a_1 = x_2$ and $y_1 + a_2 = y_2$. Thus,

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

Similarly, given $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ we have

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Ex2. Given points $A(1, 1)$ and $B(2, 3)$, draw the points A and B as well as the vector $\overrightarrow{AB}$ with initial point at A. Find the components of $\overrightarrow{AB}$.



Definitions: Given a real number λ and vectors $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$ we define vector operations component by component

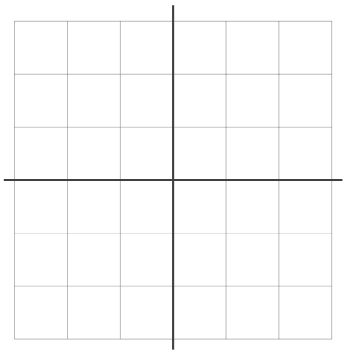
$$\lambda \vec{a} := \langle \lambda a_1, \lambda a_2 \rangle$$

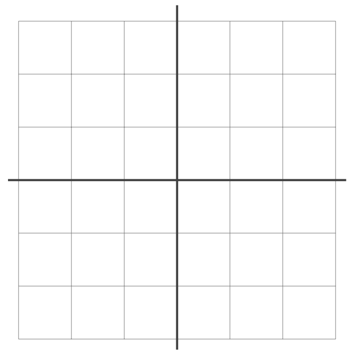
$$\vec{a} + \vec{b} := \langle a_1 + b_1, a_2 + b_2 \rangle$$

Remark: These definitions can be extended to higher dimensions.

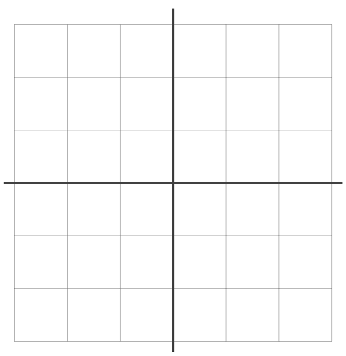
Ex3. Let $\vec{a} = \langle 3, 2 \rangle$ and $\vec{b} = \langle -1, 1 \rangle$.

- Sketch the vectors $2\vec{b}$ and $-\vec{b}$.

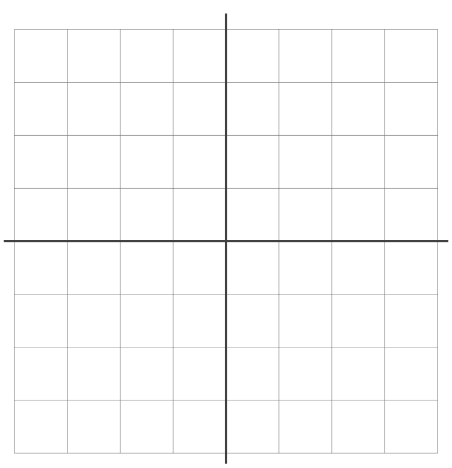




- Sketch $\vec{a}$, $\vec{b}$, and $\vec{a} + \vec{b}$.



- Sketch $\vec{a}$, $\vec{b}$, and $\vec{a} - \vec{b}$.



Notation

$|\vec{a}| = ||\vec{a}||$

DEF: The length or magnitude of a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is defined to be

$$||\vec{a}|| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}.$$

Ex4. Calculate $||\vec{a}||$ where $\vec{a} = \langle 4, 0, -3 \rangle$.

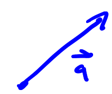
$$||\vec{a}|| = \sqrt{(4)^2 + 0^2 + (-3)^2} = \sqrt{16 + 0 + 9} = \sqrt{25} = 5$$

Ex5. Let λ be a real number. Show that $||\lambda\vec{a}|| = |\lambda|\,||\vec{a}||$

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$$\begin{aligned} ||\lambda \vec{a}|| = ||\langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle|| &= \sqrt{(\lambda a_1)^2 + (\lambda a_2)^2 + (\lambda a_3)^2} \\ &= \sqrt{\lambda^2 a_1^2 + \lambda^2 a_2^2 + \lambda^2 a_3^2} \\ &= \sqrt{\lambda^2 (a_1^2 + a_2^2 + a_3^2)} \\ &= \sqrt{\lambda^2}\,\sqrt{a_1^2 + a_2^2 + a_3^2} \\ &= |\lambda|\,||\vec{a}|| \end{aligned}$$

DEF: A unit vector $\vec{u}$ is a vector that has unit length (i.e. $||\vec{u}|| = 1$)



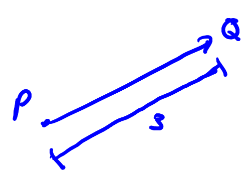
Ex6. If $\vec{a} \neq \langle 0, 0, 0 \rangle$, a unit vector in the direction of $\vec{a}$ is given by $\vec{u} = \dfrac{\vec{a}}{||\vec{a}||}$.

- Since $\frac{1}{||\vec{a}||} > 0$, $\frac{1}{||\vec{a}||}\vec{a}$ follows the same direction as $\vec{a}$.
- $||\vec{u}|| = \left|\left| \frac{\vec{a}}{||\vec{a}||} \right|\right| = \left| \frac{1}{||\vec{a}||} \right| \, ||\vec{a}|| = \frac{1}{||\vec{a}||} \, ||\vec{a}|| = 1$

Ex7. Given two points $P(1, 0, 1)$ and $Q(3, 2, 0)$, find a unit vector in the direction of $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = \langle 2, 2, -1 \rangle$

$||\overrightarrow{PQ}|| = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$



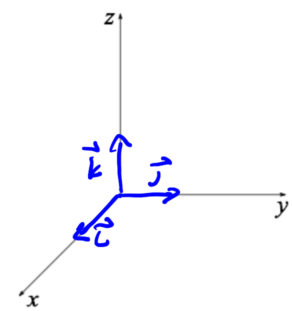
A unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{1}{||\overrightarrow{PQ}||}\overrightarrow{PQ} = \frac{1}{3}\langle 2, 2, -1 \rangle = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle$$

Ex8. The **unit** vectors **i**, **j**, and **k** given by

$$\mathbf{i} = \langle 1, 0, 0\rangle, \qquad \mathbf{j} = \langle 0, 1, 0\rangle, \qquad \mathbf{k} = \langle 0, 0, 1\rangle$$

are called the <u>standard basis vectors</u> in three dimensions. Sketch **i**, **j**, and **k**.



Ex9. Let $\vec{a} = \langle a_1, a_2, a_3\rangle$. Prove that $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

$\vec{a} = \langle a_1, a_2, a_3\rangle = \langle a_1, 0, 0\rangle + \langle 0, a_2, 0\rangle + \langle 0, 0, a_3\rangle$

$= a_1\langle 1,0,0\rangle + a_2\langle 0,1,0\rangle + a_3\langle 0,0,1\rangle$

$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

Ex10. Suppose $\vec{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\vec{b} = 4\mathbf{i} + 7\mathbf{k}$. Express the vectors $2\vec{a} + 3\vec{b}$ in terms of **i**, **j**, and **k**.

$\vec{a} = \langle 1, 2, -3\rangle$, $\vec{b} = \langle 4, 0, 7\rangle$

$2\vec{a} + 3\vec{b} = \langle 2, 4, -6\rangle + \langle 12, 0, 21\rangle$

$= \langle 14, 4, 15\rangle$ "in component form"

$\boxed{\vec{a} = 14\vec{i} + 4\vec{j} + 15\vec{k}}$

Ex11. Find a vector that has the same direction as $\vec{v} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ but has length 6.

$\vec{v} = \langle -2, 4, 2\rangle$, $\|\vec{v}\| = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$

A unit vector that follow the same direction as $\vec{v}$ is $\frac{1}{2\sqrt{6}}\langle -2, 4, 2\rangle = \frac{1}{\sqrt{6}}\langle -1, 2, 1\rangle$

The required vector is $6\left(\frac{1}{\sqrt{6}}\right)\langle -1, 2, 1\rangle = \frac{1}{\sqrt{6}}\langle -6, 12, 6\rangle$

$= \left\langle \frac{-6}{\sqrt{6}}, \frac{12}{\sqrt{6}}, \frac{6}{\sqrt{6}}\right\rangle$

Section 12.3: The Dot Product

DEF: If $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$, then the **dot product** of $\vec{a}$ and $\vec{b}$ is the number $\vec{a}\cdot\vec{b}$ given by

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

"a number"
a scalar, not a vector

Ex1. Find the dot product of the vectors $\vec{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\vec{b} = 4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$.

$\vec{a}\cdot\vec{b} = \langle 1, 2, -2\rangle \cdot \langle 4, 5, -3\rangle = (1)(4) + (2)(5) + (-2)(-3)$

$4 + 10 + 6 = 20$ "a number"

Theorem If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vectors and λ is a real number, then

dot product

(1) $\vec{a}\cdot\vec{a} = ||\vec{a}||^2$ Proof $\vec{a}\cdot\vec{a} = \langle a_1, a_2, a_3\rangle \cdot \langle a_1, a_2, a_3\rangle = a_1^2 + a_2^2 + a_3^2 = ||\vec{a}||^2$

(2) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(3) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

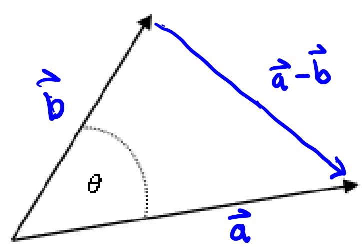
(4) $\vec{a}\cdot(\vec{b}-\vec{c}) = \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c}$

(5) $(\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\vec{b})$

(6) $\vec{0}\cdot\vec{a} = 0$

The dot product $\vec{a}\cdot\vec{b}$ can be given a geometric interpretation in terms of the angle θ between $\vec{a}$ and $\vec{b}$, which is defined to be the angle between the representations of $\vec{a}$ and $\vec{b}$ that start at the origin, where $0 \le \theta \le \pi$.

Theorem If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} = ||\vec{a}||\ ||\vec{b}||\ \cos\theta$.



Law of cosines:

$||a - b||^2 = ||\vec{a}||^2 + ||\vec{b}||^2 - 2||\vec{a}||\,||\vec{b}||\cos\theta \ldots (*)$

Note that $||\vec{a}-\vec{b}||^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$ see rule 4

$= (\vec{a}-\vec{b})\cdot\vec{a} - (\vec{a}-\vec{b})\cdot\vec{b}$

$= \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$

use $(*)$: $\cancel{\vec{a}\cdot\vec{a}} - 2\vec{a}\cdot\vec{b} + \cancel{\vec{b}\cdot\vec{b}} = \cancel{||\vec{a}||^2} + \cancel{||\vec{b}||^2} - 2||\vec{a}||\,||\vec{b}||\cos\theta$

$\vec{a}\cdot\vec{b} = ||\vec{a}||\,||\vec{b}||\cos\theta$

Ex2. Find the angle between the vectors $\vec{a} = \sqrt{3}\mathbf{i} + \mathbf{j}$ and $\vec{b} = 2\sqrt{3}\mathbf{i}$.

$\vec{a}\cdot\vec{b} = 6 + 0 + 0 = 6$

$||\vec{a}|| = \sqrt{3 + 1 + 0} = 2$

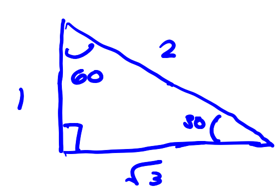
$||\vec{b}|| = \sqrt{(2\sqrt{3})^2 + 0 + 0} = 2\sqrt{3}$

We have that $\underbrace{\vec{a}\cdot\vec{b}}_{6} = \underbrace{||\vec{a}||}_{(2)}\,\underbrace{||\vec{b}||}_{(2\sqrt{3})}\,\underbrace{\cos\theta}_{\cos\theta}$

$\to \cos\theta = \dfrac{6}{2(2\sqrt{3})} = \dfrac{3}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$

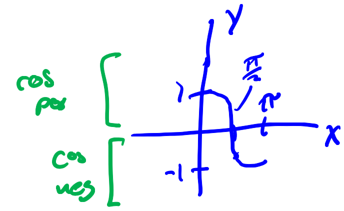
$\to \cos\theta = \dfrac{\sqrt{3}}{2}$

then $\theta = \dfrac{\pi}{6} = 30^\circ$



$1 - 2 - \sqrt{3}$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$



Dot Product and Angles

$\vec{a} \cdot \vec{b} > 0 \leftrightarrow \theta \in [0, \pi/2)$. $\qquad$ $\vec{a} \cdot \vec{b} < 0 \leftrightarrow \theta \in (\pi/2, \pi]$.

If and only if

Orthogonal Vectors: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. This is equivalent to $\vec{a} \cdot \vec{b} = 0$. The zero vector is considered to be perpendicular to all vectors. Therefore,

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

Parallel Vectors: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if $\vec{a} = \lambda\vec{b}$ for some number λ.

Ex3. Determine whether $\vec{a} = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ and $\vec{b} = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$ are orthogonal, parallel, or neither.

- $\vec{a} \cdot \vec{b} = -6 - 54 - 24 \neq 0$ so $\vec{a}$ and $\vec{b}$ are not orthogonal or perpendicular
- Is $\vec{a} = \lambda\vec{b}$ for some λ?

we want $\langle 2, 6, -4 \rangle = \lambda \langle -3, -9, 6 \rangle$ then $\begin{cases} 2 = -3\lambda \rightarrow \lambda = -2/3 \\ 6 = -9\lambda \rightarrow \lambda = -2/3 \\ -4 = 6\lambda \rightarrow \lambda = -2/3 \end{cases}$

so $\langle 2, 6, -4 \rangle = -\frac{2}{3} \langle -3, -9, 6 \rangle$

thus, $\vec{a}$ and $\vec{b}$ are parallel.

Ex4. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$ such that $\|\mathbf{u}\| = 3$, $\|\mathbf{v}\| = 4$ and the angle between these vectors is $\theta = \pi/3$.

Calculate the following:

(a) $(-2\mathbf{u}) \cdot \mathbf{u} = (-2)(u \cdot u) \overset{\text{rule 1}}{=} (-2)(\|u\|^2) = (-2)(9) = -18$

(b) $\|-3\mathbf{v}\| = |-3| \|v\| = 3(4) = 12.$

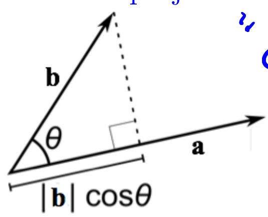
(c) $\mathbf{u} \cdot \mathbf{v} = \|u\| \|v\| \cos\theta = (3)(4)\cos\left(\frac{\pi}{3}\right) = (3)(4)\left(\frac{1}{2}\right) = 6$

$a\cdot b = |a||b|\cos\theta \rightarrow \cos\theta = \dfrac{a\cdot b}{|a||b|}$

Two Types of Projections.

The dot product is also useful to figure out the projection of a vector onto another one. We have two key concepts:

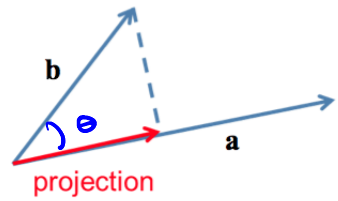
- The scalar projection of $\vec{b}$ onto $\vec{a}$:



"component b onto a"

$\mathrm{comp}_a b$

- The vector projection of $\vec{b}$ onto $\vec{a}$:



$\mathrm{comp}_a b = |b|\cos\theta = \cancel{|b|}\dfrac{a\cdot b}{|a|\cdot\cancel{|b|}} = \dfrac{a\cdot b}{|a|}$

$\mathrm{proj}_a b = |b|\cos\theta\,\dfrac{a}{|a|} = |b|\dfrac{a\cdot b}{|a|\cdot|b|}\left(\dfrac{a}{|a|}\right)$

$= \left(\dfrac{a\cdot b}{|a|^2}\right)a$

Ex5. Let $\vec{b} = 3\mathbf{i} + \sqrt{7}\mathbf{k}$ and $\vec{a} = -\mathbf{i} + \mathbf{j} + \sqrt{7}\mathbf{k}$. Find the scalar projection and the vector projection of $\vec{b}$ onto $\vec{a}$.

- $\mathrm{comp}_a b = \dfrac{a\cdot b}{|a|} = \dfrac{4}{3}$

$a\cdot b = -3+0+7=4$

$|a| = \sqrt{1+1+7} = 3$

- $\mathrm{proj}_a b = \left(\dfrac{a\cdot b}{|a|^2}\right)a$

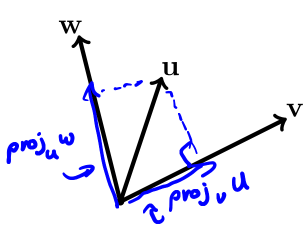
$= \left(\dfrac{4}{9}\right)\langle -1, 1, \sqrt{7}\rangle = \left\langle \dfrac{-4}{9}, \dfrac{4}{9}, \dfrac{4\sqrt{7}}{9}\right\rangle$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

Review

Ex6. Consider the following vectors in $\mathbb{R}^2$.



Which of the following is NOT correct?

(a) $proj_{\mathbf{v}}\mathbf{u}$ and $\mathbf{v}$ are parallel vectors. correct

(b) The vectors $\mathbf{u}-\mathbf{v}$ and $\mathbf{v}-\mathbf{u}$ are parallel.

$u-v = -1(u-v)$ correct

(c) $proj_{\mathbf{v}}(\mathbf{u}+\mathbf{w})$ and $\mathbf{v}$ are parallel vectors. correct

(d) $proj_{\mathbf{w}}\mathbf{u}$ and $\mathbf{u}$ are parallel vectors. incorrect